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300 ROMANIAN MATHEMATICAL CHALLENGES

ÎNVĂȚARE DE EXCELENȚĂ®
supersucces

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chapter 1 Algebra

1. For $x, y, z \geq 0$, the following inequality holds:

$$5\left(\sum\sqrt{x+y}\right)\left(\sum\sqrt{(x+y)(y+z)}\right) \geq \left(\sum\sqrt{x+y}\right)^3 + 18\sqrt{(x+y)(y+z)(z+x)}$$

where all sums are cyclic.

Leonard Giugiuc

2. Find $x \in (-1,1)$ so that $\left(M(x)\right)^n = M\left(\frac{1}{2}\right)$, where

$$M = \begin{pmatrix} \frac{1}{\sqrt{1-x^2}} & 0 & \frac{x}{\sqrt{1-x^2}} \\ 0 & 1 & 0 \\ \frac{x}{\sqrt{1-x^2}} & 0 & \frac{1}{\sqrt{1-x^2}} \end{pmatrix}.$$

Daniel Sitaru

3. Assume $1 < 2a < 2b < 2c$. Prove or disprove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} \geq \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}.$$

Daniel Sitaru

4. Consider the positive number $x_k, k = 1, \dots, n$ so that $\sum_{k=1}^{n} x_k = 1, n \geq 1$. Prove:

$$\prod_{k=1}^{n} x_k^{1/x_k} \leq \frac{1}{n^{n^2}}$$

Daniel Sitaru, Leonard Giugiuc

5. If $a, b, c \geq 1$, prove that

$$\sqrt{a^2-1} + \sqrt{b^2-1} + \sqrt{c^2-1} \leq \frac{ab+bc+ca}{2}.$$

Dorin Mărghidanu

6. Prove that if $a, b, c \in (1, \infty)$ then:

$$\sqrt[4]{\sum \frac{a}{(a-1)^2}} \geq \sqrt{6}(10-a-b-c)$$

Daniel Sitaru, Leonard Giugiuc

7. Find permutations $x, y, z \in S_5$ so that:

$$xyz^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix} z^4,$$

$$xy^2z=\begin{pmatrix}1&2&3&4&5\\1&3&2&4&5\end{pmatrix}z^2,$$
$$x^2yz=\begin{pmatrix}1&2&3&4&5\\5&3&2&4&1\end{pmatrix}z^2.$$

Daniel Sitaru

8. Let x,y,a,b be real numbers so that $(x+1)^2+y^2=1$ and $(a-2)^2+b^2=4$. Find all possible values of $ax+by$.

Leonard Giugiuc

9. Consider $x,y,z,t\in(0,1),\Omega=\sum_{cycl}\sqrt{x^2+(1-y)^2}$.
Find $m=\inf\Omega$ and $M=\sup\Omega$.

Daniel Sitaru

10. Consider the real numbers $a>0$ and $b\in\left(\frac{a}{2},a\right]$. Prove that:

a. $\dfrac{1}{x^2-ax+b^2}\le\dfrac{x+2b-a}{(2b-a)b^2}$, for any $x\in[0,\infty)$;

b. $\dfrac{1}{x^2-ax+b^2}+\dfrac{1}{y^2-ay+b^2}+\dfrac{1}{z^2-az+b^2}\le\dfrac{3}{(2b-a)b}$,

for any non-negative x, y, z, satisfying $x+y+z=3(a-b)$. When does the equality hold?

Dan Popescu

11. Consider the strictly positive real numbers a,b,c,d with the property $ab+bc+cd+da\le 8$. Prove that:

$$\frac{a^2+b^2}{(a+b)^4}+\frac{b^2+c^2}{(b+c)^4}+\frac{c^2+d^2}{(c+d)^4}+\frac{d^2+a^2}{(d+a)^4}\le\frac{1}{abcd}.$$

Traian Tămîian

12. Given the real numbers $a,b\in(1,\infty),a<b$ and numbers $x,y\in(a,b)$, prove that:

$$\log_x[(a+b)y-ab]\cdot\log_y[(a+b)x-ab]\ge 4.$$

Ion Călinescu

13. Given $m,n\in\mathbb{N},m,n\ge 2$ and $x_1,x_2,x_3,x_4\in[0,\infty)$ so that

$$x_1+x_2+x_3+x_4=4.$$

a. Prove that $\sqrt[m]{x_1}\cdot\sqrt[n]{x_2}+\sqrt[m]{x_2}\cdot\sqrt[n]{x_3}+\sqrt[m]{x_3}\cdot\sqrt[n]{x_4}+\sqrt[m]{x_4}\cdot\sqrt[n]{x_1}\le 4$.

b. When does the equality hold in a.?

Gheorghe Alexe

14. Solve for real number x: $2^x+1=(3^x-1)^{\log_2 3}$.

Eugen Radu

15. Solve for real numbers x,y,z the following system:

$$\begin{cases}\sqrt[3]{x}+2\sqrt[6]{yz}=13\\\sqrt[3]{y}+2\sqrt[6]{xz}=13\\\sqrt[3]{z}+2\sqrt[6]{xy}=13\end{cases}.$$

Nelu Chichirim

solutions 1 | Algebra

1. Solution (*The Romanian Crew – Claudia Nănuţi, Diana Trăilescu, Daniel Sitaru, Leonard Giugiuc*)
If $(x+y)(y+z)(z+x)=0$, then at least two of x, y, z vanish so that the inequality reduces to $10a\sqrt{a} \geq 8a\sqrt{a}$, for some $a \geq 0$, which is obviously true. Assume $(x+y)(y+z)(z+x) \neq 0$. Then $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ form the sides of a triangle. The segments of the sides from the vertices of the triangle to the points of tangency with the incircle split each of the sides into two parts, thus insuring the existence of $a, b, c > 0$ so that $\sqrt{x+y} = a+b, \sqrt{y+z} =$ $= b+c, \sqrt{z+x} = c+a$. In terms of a, b, c the inequality becomes

$$5(a+b+c)[a^2+b^2+c^2+3(ab+bc+ca)] \geq$$
$$\geq 4(a+b+c)^3 + 9(a+b)(b+c)(c+a).$$

Introduce $S = a^3+b^3+c^3, s = ab(a+b)+bc(b+c)+ca(c+a)$and $p = abc$. The straightforward algebraic manipulation reduces the above inequality to $5S+5s+15s+45p \geq 4S+12s+24p+9s+18p$ which is simplified to $S+3p \geq s$. The latter is one of the particular cases of Schur's inequality.

2. Solution 1 (*Leonard Giugiuc*)
Let $a = \frac{1}{\sqrt{1-x^2}}$ and $= \frac{x}{\sqrt{1-x^2}}$.
It is left as an exercise to prove by mathematical induction that

$$\begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix}^n = \begin{pmatrix} n & 0 & b_n \\ 0 & 1 & 0 \\ b_n & 0 & a_n \end{pmatrix},$$

where $a_n = \frac{1}{2}[(a+b)^n + (a-b)^n]$ and $b_n = \frac{1}{2}[(a+b)^n - (a-b)^n]$. In our case, $a_n = \frac{1}{2}\left[\sqrt{\left(\frac{1+x}{1-x}\right)^n} + \sqrt{\left(\frac{1-x}{1+x}\right)^n}\right]$ and $b_n = \frac{1}{2}\left[\sqrt{\left(\frac{1-x}{1+x}\right)^n} + \sqrt{\left(\frac{1+x}{1-x}\right)^n}\right]$.

We need to solve the equations $a_n = \frac{2}{\sqrt{3}}$ and $b_n = \frac{1}{\sqrt{3}}$. Define

$$t = \sqrt{\left(\frac{1-x}{1+x}\right)^n} > 0.$$