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MATH PHENOMENON

ÎNVĂȚARE DE EXCELENȚĂ®
supersucces

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Chapter 1 Famous Inequalities

Cauchy–Schwarz Inequality

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 < \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

Minkowski Inequality

$$\left(\sum\nolimits_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum\nolimits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum\nolimits_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}; \text{ for } p \ge 1.$$

Hölder's Inequality

$$\sum\nolimits_{i=1}^{n} |x_i y_i| \le \left(\sum\nolimits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum\nolimits_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}} \text{ for } p, q, > 1, \ \frac{1}{p} + \frac{1}{q} = 1.$$

Bernoulli Inequality

$(1 + x)^r \ge 1 + rx$ for $x \ge 1$, $r \in \mathbb{R} \setminus (0, 1)$. Reverse for $r \in [0, 1]$.

$(1 + x)^r \le 1 + (2^r - 1)x$ for $x \in [0, 1]$, $r \in \mathbb{R} \setminus (0, 1)$.

$(1 + x)^n \le \dfrac{1}{1 - nx}$ for $x \in [-1, 0]$, $n \in \mathbb{N}$.

$(1 + x)^r \le 1 + \dfrac{rx}{1 - (r - 1)x}$ for $x \in \left[-1, \dfrac{1}{r - 1}\right)$, $r > 1$.

$(1 + nx)^{n+1} \ge (1 + (n + 1)x)^n$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$.

$(a + b)^n \le a^n nb(a + b)^{n-1}$ for $a, b \ge 0$, $n \in \mathbb{N}$.

$\left(1 + \dfrac{x}{p}\right)^p \ge \left(1 + \dfrac{x}{q}\right)^q$ for (i) $x > 0$, $p > q > 0$,

(ii) $-p < -q < x < 0$, (iii) $-q > -p > x > 0$. Reverse for:
(iv) $q < 0 < p$, $-p > x > 0$, (v) $q < 0 < p$, $-p < x < 0$.

Exponential Inequalities

$e^x \ge \left(1 + \dfrac{x}{n}\right)^n \ge 1 + x$, $\left(1 + \dfrac{x}{n}\right)^n \ge e^x\left(1 - \dfrac{x^2}{n}\right)$ for $n > 1$, $|x| \le n$.

$e^x \ge x^e$ for $x \in \mathbb{R}$, and $\dfrac{x^n}{n!} + 1 \le e^x \le \left(1 + \dfrac{x}{n}\right)^{n + \frac{x}{2}}$ for $x, n > 0$.

$e^x \geq 1 + x + \frac{x^2}{2}$ for $x \geq 0$, reverse for $x \leq 0$.

$e^{-x} \leq 1 - \frac{x}{2}$ for $x \in [0, \sim 1.59]$ and $2^{-x} \leq 1 - \frac{x}{2}$ for $x \in [0, 1]$.

$\frac{1}{2-x} < x^x < x^2 - x + 1$ for $x \in (0, 1)$.

$x^{\frac{1}{r}}(x-1) \leq rx\left(x^{\frac{1}{r}} - 1\right)$ for $x, r \geq 1$.

$x^y + y^x > 1$ and $e^x > \left(1+\frac{x}{y}\right)^y > e^{\frac{xy}{x+y}}$ for $x, y > 0$.

$2 - y - x^{-x-y} \leq 1 + x \leq y + e^{x-y}$, and $e^x \leq x + e^{x^2}$ for $x, y \in \mathbb{R}$.

Logarithm Inequalities

$\frac{x-1}{x} \leq \ln(x) \leq \frac{x^2-1}{2x} \leq x - 1$, $\ln(x) \leq n\left(x^{\frac{1}{n}} - 1\right)$ for $x, n > 0$.

$\frac{2x}{2+x} \leq \ln(1+x) \leq \frac{x}{\sqrt{x+1}}$ for $x \geq 0$, reverse for $x \in (-1, 0]$.

$\ln(n+1) < \ln(n) + \frac{1}{n} \leq \sum_{i=1}^{n}\frac{1}{i} \leq \ln(n) + 1$.

$\ln(1+x) \geq \frac{x}{2}$ for $x \in [0, \sim 2.51]$, reverse elsewhere.

$\ln(1+x) \geq x - \frac{x^2}{2} + \frac{x^3}{4}$ for $x \in [0, \sim 0.45]$, reverse elsewhere.

$\ln(1-x) \geq -x - \frac{x^2}{2} + \frac{x^3}{2}$ for $x \in [0, \sim 0.43]$, reverse elsewhere.

Trigonometric Inequalities

$x - \frac{x^3}{2} \leq x\cos x \leq \frac{x\cos x}{1-\frac{x^2}{3}} \leq x\sqrt[3]{\cos x} \leq x - \frac{x^3}{6} \leq x\cos\frac{x}{\sqrt{3}} \leq \sin x$.

Hyperbolic Inequalities

$x\cos x \leq \frac{x^3}{\sinh^2 x} \leq x\cos^2\left(\frac{x}{2}\right) \leq \sin x \leq \frac{(x\cos x + 2x)}{3} \leq \frac{x^2}{\sinh x'}$.

$\frac{2}{\pi}x \leq \sin x \leq x\cos\left(\frac{x}{2}\right) \leq x \leq x + \frac{x^3}{3} \leq \tan x$ all for $x \in \left[0, \frac{\pi}{2}\right]$.

$\cosh(x) + \alpha\sinh(x) \leq e^{x\left(\alpha+\frac{x}{2}\right)}$ for $x \in \mathbb{R}$, $\alpha \in [-1, 1]$.

Chapter 2 Algebra

A.1. If $a, b, c, d \in (0, \infty)$, then: $\dfrac{a}{a+2b}+\dfrac{b}{2a+b}+\dfrac{c}{c+2d}+\dfrac{d}{2c+d}\ge\dfrac{4}{3}$.

A.2. If $a, b, c \in \mathbb{N}^* \setminus \{1\}$, then: $\left(1-\dfrac{1}{a^2}\right)\left(1-\dfrac{1}{b^2}\right)\left(1-\dfrac{1}{c^2}\right)>\dfrac{1}{8}$.

A.3. If $x_1, x_2, \ldots, x_n$ are positive real numbers; $n \in \mathbb{N}^*$ and $4^n \cdot x_1x_2 \ldots x_n = 1$, prove that:

$$4^n(x_1^2+x_2)(x_2^2+x_3)\cdot\ldots\cdot(x_{n-1}^2+x_n)(x_n^2+x_1)\ge 1.$$

A.4. If $x_1, x_2, \ldots, x_{2n+1} \in \mathbb{Z}$; $n \in \mathbb{N}^*$; $x_1x_2 \cdot \ldots \cdot x_{2n+1} = 1$ then find $x_1, x_2, \ldots, x_{2n+1}$ so that:

$$|x_1 - x_2| = |x_2 - x_3| = \ldots = |x_{2n} - x_{2n+1}| = |x_{2n+1} - x_1|.$$

A.5. Prove that: $\left(1+\dfrac{\pi}{e}\right)^9+\left(1+\dfrac{e}{\pi}\right)^9\ge 1024$.

A.6. If $x, y \in \mathbb{Z}$; $|x| \le 11$; $|y| \le 100$, prove that: $\left|x\sqrt{5}+y\sqrt{7}\right|>\dfrac{1}{133}$.

A.7. If $x, y, z \in (0, \infty)$ prove that:

$$\frac{x}{(1+x)^2}+\frac{y}{(1+x+y)^2}+\frac{z}{(1+x+y+z)^2}\le\frac{x+y+z}{1+x+y+z}.$$

A.8. Prove that, for any $x, y, z \in (0, \infty)$.

$$x^3+y^3+z^3+xyz(xy+yz+xz)\ge 8\left(\frac{y^2z^2\sqrt{x}}{(z+y)^2}+\frac{x^2y^2\sqrt{z}}{(x+y)^2}+\frac{z^2x^2\sqrt{y}}{(z+x)^2}\right).$$

A.9. Prove that, for any $x, y, z \in (0, \infty)$.

$$x^3+y^3+z^3+x+y+z\ge 4\left(\frac{x^2y}{x+y}+\frac{y^2z}{y+z}+\frac{z^2x}{z+x}\right).$$

A.10. Prove that in triangle ABC the following relation is valid:

$$\sum\sqrt{p+\sqrt{ab+ac}}\le\sqrt{2}\sum\sqrt{a+b}.$$

A.11. If $a, b, c \in [2, 4]$ then: $(a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)\le\dfrac{81}{8}$.

A.12. Solve the system for real numbers:

$$\begin{cases} x_2 = \sqrt{\dfrac{x_1^2+1}{2}} \\ x_3 = \sqrt{\dfrac{x_2^2+1}{2}} \\ ------; n \in \mathbb{N}; n \geq 3 \\ x_n = \sqrt{\dfrac{x_{n-1}^2+1}{2}} \\ x_1 = \sqrt{\dfrac{x_n^2+1}{2}} \end{cases}$$

A.13. Prove that if $a, b, c \in (0, \infty)$, $abc = 1$, then:

$$a^3(b^2 + c) + c^3(a^2 + b) + b^3(c^2 + a) \geq 6.$$

Generalization: Prove that if $a, b, c \in (0, \infty)$, $abc = x$ then:

$$a^3(b^2 + c) + c^3(a^2 + b) + b^3(c^2 + a) \geq x(8\sqrt{x} - x - 1).$$

A.14. Prove that if $x, y, z \in [1, \infty)$ then:

$$\sum_{cyc}(x+y)\sqrt{z-1} \leq xy + yz + zx.$$

A.15. Find the natural numbers (integers) a, b, c so that:

$$(1 + bc)(1 + ac)(1 + ba) = (1 + a)(1 + b)(1 + c)$$

A.16. Prove that for any $a, b, c \in (0, \infty)$; $a \neq b \neq c \neq a$.

$$\sum_{cyc}\left(a + \frac{2ab}{a+b} - 2\sqrt{ab}\right) < \sum_{cyc}|a-b|$$

A.17. Prove that, if $1 \leq x \leq y < z \leq \dfrac{3}{2}$ then:

$$(x+y)\sqrt{3-2z} + (y+z)\sqrt{3-2x} + (z+x)\sqrt{3-2y} < 2(xy + yz + xz).$$

A.18. Determine $x, y, z \in (0, \infty)$ so that: $\begin{cases}(x+y)(1+z) = 20 \\ xyz = 25\end{cases}$

A.19. If $a, b, c \in (0, \infty)$ then:

$$\sum\left(a + \frac{b}{a^2}\right)\left(1 + \frac{a}{b}\right)c \geq 4(a+b+c).$$

A.20. If $m \in \mathbb{N}$; $m \geq 2$; m – fixed and $x, y, z \in (0, \infty)$, find the minimum of the sum

$$S = \sum \frac{x^2}{(my+z)(mz+y)}.$$