10 Years of Romanian Mathematical Competitions

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Preface

In 1997, Mircea Becheanu and Bogdan Enescu, then strongly involved in the training and participation of the Romanian teams in the International Mathematical Olympiads had the idea to produce an annual booklet that should contain the problems given at the National Mathematical Olympiads and at the Romanian selection tests. Every year since, the book was offered to the leaders and deputy leaders at the IMOs.

It seemed that *Romanian Mathematical Competitions*, the title chosen for the booklet, became in years a great success. More than 100 problems are completely discussed and more than 60 authors are present. The Editors of the booklet are known mathematicians in the problem solving domain in Romania.

The 59th IMO, to be held in Cluj-Napoca this year, gave us the opportunity to celebrate the last 10 years of the RMC book with a selection of the nicest problem used in competitions in the last 10 years, problems that were proposed by Romanian authors. We decided to categorize the problems following the IMO-shortlist: algebra, combinatorics, geometry and number theory. Given that the terminal classes in the Romanian Curriculum study analysis and abstract algebra, we invented the Putnam category.

Many thanks are due to all people involved in the 20 years of the RMC. But mostly we are grateful to Mircea Becheanu, Călin Popescu, Marean Andronache, Dinu Şerbănescu, Barbu Berceanu, Mihai Bălună, Bogdan Enescu and of course our great friend and talented mathematician who was Dan Schwarz. The work of Alex Negrescu completed the LaTeX version of the last years' editions.

We strongly hope that this book will be useful for many countries in training their gifted students for the International Mathematical Olympiads.

Radu Gologan
President of the Romanian Mathematical Society

Algebra

1.1. Problems

A1. We say that the function $f : \mathbb{Q}_+^* \to \mathbb{Q}$ has property $\mathcal{P}$ if

$$f(xy) = f(x) + f(y),$$

for any $x, y \in \mathbb{Q}_+^*$.

a) Prove that there are no one-to-one functions with property $\mathcal{P}$;

b) Does there exist onto functions with property $\mathcal{P}$?

Bogdan Moldovan - ONM 2017

A2. Determine all integers $n \geq 2$ such that $a + \sqrt{2} \in \mathbb{Q}$ and $a^n + \sqrt{2} \in \mathbb{Q}$, for some real number a depending on n.

Mihai Bălună - tIMO 2017

A3. Let $a \in \mathbb{N}, a \geqslant 2$. Prove that the following statements are equivalents:

a) One can find $b, c \in \mathbb{N}^*$ such that $a^2 = b^2 + c^2$;

b) One can find $d \in \mathbb{N}^*$, such that the equations $x^2 - ax + d = 0$ and $x^2 - ax - d = 0$ have integer roots.

Vasile Pop - OJM 2016

A4. **a)** Prove that there exist non-periodical functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(x+1\right) + f\left(x-1\right) = \sqrt{5} f\left(x\right),$$

for all $x \in \mathbb{R}$;

b) Prove that any function $g : \mathbb{R} \to \mathbb{R}$ such that

$$g\left(x+1\right) + g\left(x-1\right) = \sqrt{3} g\left(x\right),$$

for all $x \in \mathbb{R}$, is periodical.

Vasile Pop - OJM 2016

A5. Let $f:\mathbb{R}\to\mathbb{R}$ be a function with the following properties:

$$(P1): \qquad f(x+y) \leqslant f(x)+f(y)$$

and

$$(P2): \qquad f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y),$$

for all $x, y \in \mathbb{R}$ and $t \in [0,1]$.

a) Prove that

$$f(b)+f(c) \leqslant f(a)+f(d),$$

for any $a \leqslant b \leqslant c \leqslant d$ such that $d-c=b-a$;

b) Prove that

$$f\left(\sum_{k=1}^{n} x_k\right)+(n-2)\left(\sum_{k=1}^{n} f(x_k)\right) \geqslant \sum_{1\leqslant i<j\leqslant n} f(x_i+x_j),$$

for all $n\in\mathbb{N}, n\geqslant 3$, and $x_1, x_2, ..., x_n \in \mathbb{R}$.

Nicolae Bourbăcuţ - ONM 2016

A6. Determine all functions $f:\mathbb{N}^*\to\mathbb{N}^*$, such that $f(m)\geqslant m$ and $f(m+n)$ divides $f(m)+f(n)$, for all $m,n\in\mathbb{N}^*$.

Marius Cavachi - tIMO 2016

A7. Find all real numbers a and b such that the equality

$$\lfloor ax+by\rfloor+\lfloor bx+ay\rfloor=(a+b)\lfloor x+y\rfloor$$

holds, for every $x,\ y\in\mathbb{R}$.

Lucian Dragomir - OJM 2015

A8. A quadratic function f sends any interval I of length 1 to an interval $f(I)$ of length at least 1. Prove that for any interval J of length 2, the length of the interval $f(J)$ is at least 4.

Mihai Bălună - ONM 2015

A9. Let a be a natural odd number which is not a perfect square. If $m, n \in \mathbb{N}^*$ prove that:

a) $\{m(a+\sqrt{a})\} \neq \{n(a-\sqrt{a})\}$;

b) $\lfloor m(a+\sqrt{a}) \rfloor \neq \lfloor n(a-\sqrt{a}) \rfloor$.

Vasile Pop - ONM 2014

A10. Consider the function $f : \mathbb{N}^* \to \mathbb{N}^*$ which satisfies the properties:

a) $f(1) = 1$;

b) $f(p) = 1 + f(p-1)$, for any prime number p;

c) $f(p_1 p_2 \cdots p_n) = f(p_1) + f(p_2) + \cdots + f(p_n)$, for any prime numbers $p_1, p_2, ..., p_n$, not necessary distinct.

Prove that $2^{f(n)} \leq n^3 \leq 3^{f(n)}$, for any $n \in \mathbb{N}$, $n \geq 2$.

George Stoica - ONM 2014

A11. Find all functions $f, g : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(g(x) + g(y)) = f(g(x)) + y$$

and

$$g(f(x) + f(y)) = g(f(x)) + y,$$

for all $x, y \in \mathbb{Q}$.

Vasile Pop - ONM 2015

A12. Let $n \in \mathbb{N}, n \geq 2$. Prove that there exist $n+1$ pairwise distinct numbers $x_1, x_2, ..., x_n, x_{n+1} \in \mathbb{Q} \setminus \mathbb{Z}$ such that

$$\{x_1^3\} + \{x_2^3\} + \cdots + \{x_n^3\} = \{x_{n+1}^3\},$$

where $\{x\}$ denotes the fractional part of the real number x.

Dorel Miheţ - tIMO 2014

A13. Given an odd prime p, determine all polynomials f and g, with integral coefficients, satisfying the condition $f(g(X)) = \sum_{k=0}^{p-1} X^k$.

Cezar Lupu & Vlad Matei - tIMO 2014

1.2. Solutions

A1. a) For $x = y = 1$ we obtain $f(1) = 0$. For $x = y$ we get $f(x^2) = 2f(x)$, and it is not difficult to prove that $f(x^n) = nf(x)$, for all $n \in \mathbb{Z}$ and $x \in \mathbb{Q}_+^*$.

Let p and q be two distinct prime numbers. We could find $a, b, c \in \mathbb{Z}^*$ such that $f(p) = \frac{a}{c}$ and $f(q) = \frac{b}{c}$. Then

$$f(p)f(q) = f(p) \cdot \frac{b}{c} = \frac{f(p^b)}{c} = \frac{a}{c} \cdot f(q) = \frac{f(q^a)}{c}.$$

If f were one-to-one, then $p^b = q^a$, and thus $a = b = 0$, which is a contradiction.

b) If $x > 0$ is a rational number, we can uniquely find the distinct prime numbers $p_1, p_2, ..., p_n$ and $a_1, a_2, \ldots, a_n \in \mathbb{N}^*$ such that $x = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$. It is easy to check that

$$f(x) = a_1 f(p_1) + a_2 f(p_2) + \ldots + a_n f(p_n),$$

so it will be enough to define the function f on the set of prime numbers. Define the function f by

$$f(p) = \frac{1}{p!},$$

for any prime number p. Then, for any $a \in \mathbb{Z}$, we have $f(p^a) = \frac{a}{p!}$. We prove that f is an onto function. Let $y \in \mathbb{Q}$, $y = \frac{a}{b}$, with $a, b \in \mathbb{Z}$. Choose a prime number p, large enough such that b divides $p!$. Then $p!y = p!\frac{a}{b} \in \mathbb{Z}$, and thus

$$f(p^{p!y}) = \frac{p!y}{p!} = y.$$

A2. There is only one such n, namely $n = 2$, in which case we may take

$$a = \frac{1}{2} - \sqrt{2}.$$

Verification is routine and hence omitted.