Marius Drăgan

INEQUALITIES IN BICENTRIC QUADRILATERAL

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Preface

The work that we review is a creation of prof. Marius Drăgan from Technical College "Mircea cel Bătrân" in Bucharest, well known for his preoccupation related to the domain of inequalities, author of several books, articles and interesting problems.

The book is structured into 9 chapters; some of them are divided into subchapters. The first chapter contains introductive notions and some metrical relations, among which we can mention the Durrande-Fuss relation which expresses the distance between the centers of the inscribed and circumscribed circle according to their radius.

The second chapter approaches the Blundon-Eddy inequality in a bicentric quadrilateral (PB), analogous to the Blundon's inequality in a triangle, as well as PB inequality.

The third chapter is dedicated to Hadwiger-Finsler inequality in PB, having two types for this, plus the reverse of the first type, at the same time demonstrating an analogous in PB of the Ionescu-Weizenbock inequality.

Chapter 4 contains the inequality of Yun and refinements of it (some obtained by the method of mathematical analysis) and other considerations on the cyclical sum of general term $\sqrt{\sin \frac{A}{2} \cos \frac{B}{2}}$.

Chapter 5 undertakes a study of the length of the tangents from the vertexes to the inscribed circle, perceived as the roots of the 4^{th} degree equations, then the Durrande-Fuss relation is being demonstrated and other inequalities.

Chapter 6 establishes inequalities with exponents similar to those of the sides, based on the the monotony study of functions.

Chapter 7 revisits the Blundon-Eddy inequality with the strong methods of the analysis, and chapter 8 undertakes a similar study to the one in chapter 6, only this time for exponents similar to the length of the tangents. Finally, other interesting inequalities appear in chapter 9, the refinement of the Fejes Tóth $R \geq \sqrt{2}\,r$ inequality. For an easier acknowledgement of references, the author places a specific bibliography after each chapter and subchapter, besides the general bibliography at the end of the book.

We appreciate the difficulty of the book which succeeds to systematize results from the quantitative geometry (the relation between lengths, surfaces) of PB, many of them of recent date, some pertaining to the author.

Given the technical difficulties, the presentation is clear and concise, although some of the supplementary comments would be welcomed.

Through this paper, the author proves he is a specialist in the domain of inequalities, particularly the geometrical ones, but not only.

We kindly recommend the publication of this elegant book, at the border between geometry, algebra and mathematical analysis.

Prof. dr. **Marcel Ţena**

Chapter 1

Preliminary notions

Definition 1.1 A bicentric quadrilateral is a convex quadrilateral that has both an incircle and a circumcircle. The radius of these circles are called circumradius and irradius. We denote their lengths with R and r, with O the center of circumcircle and I the center of incircle. Bicentric quadrilateral respects all the properties of tangential and cyclic quadrilaterals.

Poncelet porism. If two circles one within the other are the incircle and circumcircle of a bicentric quadrilateral, then all the points of circumcircle are the vertices of a bicentric quadrilateral, having the same incenter and circumcircle.

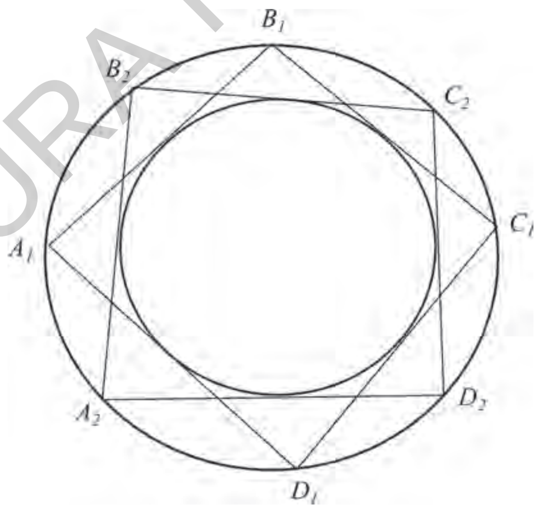

Figure 1.1

If we denote with $\overline{d} = OI$ and if $\overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2 + r^2}$ (Durrande-Fuss), then this is a sufficient condition to exist a quadrilateral with the incenter $\mathcal{C}(I, r)$ and the circumcenter $\mathcal{C}(O, R)$.
We denote $t_1 = AM$, $t_2 = MB$, $t_3 = CP$, $t_4 = DQ$.
We have $UM^2 + (MI - OU)^2 = \overline{d}^2$ or

$$\frac{(t_1 - t_2)^2}{t_1} + r^2 - 2rOU + R^2 - \frac{(t_1 + t_2)^2}{4} = R^2 + r^2 - r\sqrt{4R^2 + r^2}$$

or

$$r\sqrt{4R^2 + r^2} - t_1 t_2 = r\sqrt{4R^2 - (t_1 + t_2)^2}.$$

After squaring we obtain

$$2rt_1t_2\sqrt{4R^2 + r^2} = r^4 + r^2(t_1 + t_2)^2 + t_1^2 t_2^2.$$

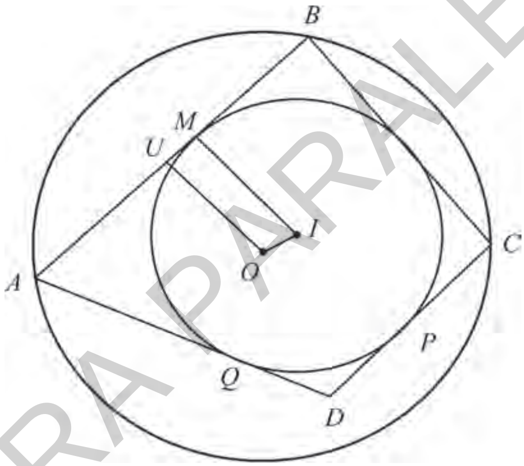

Figure 1.2

Similarly

$$2rt_2t_3\sqrt{4R^2 + r^2} = r^4 + r^2(t_2 + t_3)^2 + t_2^2 t_3^2.$$

After we divided the two equalities and perform some calculation, we obtain

$$r^4 + (t_2^2 - t_1 t_3)r^2 - t_1 t_2^2 t_3 = 0$$

or

$$(r^2 - t_1 t_3)(r^2 + t_2^2) = 0$$

or $\quad t_1 t_3 = r^2$. So $D \in \mathcal{C}(O, R)$.

In the following we denote by $a = AB$, $b = BC$, $c = CD$, $d = DA$, $\overline{d} = OI$, s the semiperimeter, R the radius, r the inradius, F the area, $x_1 = bc + ad$, $x_2 = ab + cd$, $x_3 = ac + bd$, $BD = d_1$, $AC = d_2$. We have the following lemma:

Lemma 1.1 *In every bicentric quadrilateral the following equalities are true:*

$$F^2 = (s-a)(s-b)(s-c)(s-d) = abcd \tag{1.1}$$

$$x_1 + x_2 = s^2 \tag{1.2}$$

$$x_1 = \frac{4sRr}{d_1}, \quad x_2 = \frac{4sRr}{d_2} \tag{1.3}$$

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{s}{4Rr} \tag{1.4}$$

$$x_1x_2x_3 = 16R^2r^2s^2 \tag{1.5}$$

$$x_3 = 2r\left(\sqrt{4R^2+r^2}+r\right) \tag{1.6}$$

$$\begin{aligned}&(x_1-x_2)^2(x_2-x_3)^2(x_3-x_1)^2\\ &= (a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2\end{aligned} \tag{1.7}$$

$$\begin{aligned}&(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2\\ &= 16r^4s^2\left[s^2-8r\left(\sqrt{4R^2+r^2}-r\right)\right]\left[s^2-\left(r+\sqrt{4R^2+r^2}\right)\right]^2\end{aligned} \tag{1.8}$$

$$d_1 + d_2 = \frac{s}{2R}\left(\sqrt{4R^2+r}+r\right) \tag{1.9}$$

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ad+bc)(ac+bd)}{abcd}} \tag{1.10}$$

$$r = \frac{\sqrt{abcd}}{a+c} = \frac{\sqrt{abcd}}{a+b} \tag{1.11}$$

$$\cos A = \frac{bc-ad}{bc+ad} \tag{1.12}$$

$$\sin\frac{A}{2} = \sqrt{\frac{ad}{ad+bc}} \tag{1.13}$$

$$\cos\frac{A}{2} = \sqrt{\frac{bc}{ad+bc}} \tag{1.14}$$

$$\tan\frac{A}{2} = \sqrt{\frac{ad}{bc}}\,. \tag{1.15}$$

Chapter 2

Blundon-Eddy inequality

2.1 A geometrical proof of Blundon-Eddy inequality in bicentric quadrilateral

The purpose of this chapter is to give a geometrical proof of Blundon-Eddy inequality using basic knowledge of mathematical analysis.

Let $ABCD$ a bicentric quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, O the center of circumcenter with radius R, I the center of incenter with radius r, S the semiperimeter, $\overline{d} = OI$, $x_1 = bc + ad$, $x_2 = ab + cd$, $x_3 = ac + bd$. We remember the following lemma.

Lemma 2.1 *In every bicentric quadrilateral the following equalities are true:*

i) $x_1 + x_2 = S^2$

ii) $x_1 x_2 x_3 = S^2 + 2r^2 + 2r\sqrt{4R^2 + r^2}$

iii) $x_3 = 2r\left(\sqrt{4R^2 + r^2} + r\right)$

iv) $\tan^2 \dfrac{A}{2} = \dfrac{bc}{ad}$

Proof. *i), ii)* and *iii)* are proved in [2], *iv)* see [6].

Also, the Fuss theorem is well known.

Theorem 2.1 *In every bicentric quadrilateral the following equalities are true:*

i) $\overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2 + r^2}$

ii) $\dfrac{1}{(R - \overline{d})^2} + \dfrac{1}{(R + \overline{d})^2} = \dfrac{1}{r^2}$

Proof. See [3], [4] and [5].

In the following we give a new proof of Blundon-Eddy inequality.

Theorem 2.2 *In every bicentric quadrilateral $ABCD$ the inequality $S_1 \le S \le S_2$ is true, where $S_1 = \sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$ represents the semiperimeter of bicentric quadrilateral $A_1B_1C_1D_1$ with the sides:*

$$\{a_1, b_1, c_1, d_1\} = \Big\{2\sqrt{R^2-(r+\overline{d})^2},\ \sqrt{R^2-(r-\overline{d})^2}+\sqrt{R^2-(r+\overline{d})^2},\ 2\sqrt{R^2-(r-\overline{d})^2},\ \sqrt{R^2-(r-\overline{d})^2}+\sqrt{R^2-(r+\overline{d})^2}\Big\}$$

and $S_2 = \sqrt{4R^2+r^2}+r$ the semiperimeter of bicentric quadrilateral $A_2B_2C_2D_2$ with the sides

$$\{a_2, b_2, c_2, d_2\} = \left\{\frac{2R}{R-\overline{d}}\sqrt{(R-\overline{d})^2-r^2},\ \frac{2R}{R+\overline{d}}\sqrt{(R+\overline{d})^2-r^2},\ \frac{2R}{R+d}\sqrt{(R+\overline{d})^2-r^2},\ \frac{2R}{R-\overline{d}}\sqrt{(R-\overline{d})^2-r^2}\right\}$$

Proof. Let $\mathcal{C}(O,R)$, $\mathcal{C}(I,r)$ with R, r fixed and S variable which satisfies the Fuss theorem.
We consider the vertical diameter A_2C_2 such that A_2, I, O, C_2 be collinear.

Let $A \in \mathcal{C}(O,R)$ a variable point situated on the left semiplane in relation to A_2C_2 and $B, C, D \in \mathcal{C}(O,R)$ such that AB, BC, CD are tangent to $\mathcal{C}(I,r)$.

According to Poncelet theorem, A_2 is tangent to $\mathcal{C}(I,r)$.

We denote $\alpha = \mu(\widehat{AOA_2})$, $\alpha \in [0,\pi]$

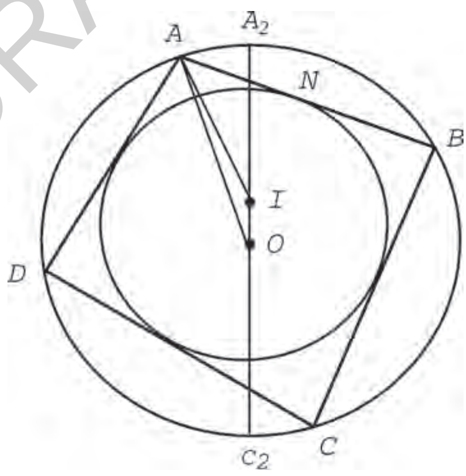

Figure 2.1